

SOLUTION-GENERATING TECHNIQUES

For stationary axisymmetric space-times many solution-generating techniques are known. These are associated with the two Killing vectors, and with the internal symmetries of the Ernst equation. Colliding plane wave solutions also have two Killing vectors, and the main field equations can be transformed exactly to the Ernst equation. The main difference is that the solutions of the Ernst equation now contain the metric functions explicitly rather than the field potentials as in the stationary axisymmetric case. It is therefore to be expected that many of the familiar solution generating techniques can be used to obtain new colliding plane wave solutions from already known, or ‘seed’, solutions.

In this chapter we will consider only vacuum solutions. This will enable us to continue to concentrate on the collision of gravitational waves. Various techniques will be discussed here, and the main solutions that have been obtained using them will be described in the next chapter.

Many techniques are also known by which solutions of the Einstein–Maxwell equations, and other non-vacuum solutions, can be generated from known vacuum solutions. These will be discussed later in Chapter 15.

12.1 The colinear case

Consider first the case when the approaching waves have colinear polarization. The metric can be taken in the form of the line element (6.20) with $W = 0$. In this case, given any solution U_o , V_o , M_o of the field equations (6.22), then another solution of the same equations again with $W = 0$ is given by U , V and M , where

$$\begin{aligned} U &= U_o = -\log(f(u) + g(v)) \\ V &= V_o + aU_o \\ M &= M_o + aV_o + \frac{1}{2}a^2U_o \end{aligned} \tag{12.1}$$

where a is an arbitrary constant.

This transformation is well known (Halilsoy, 1985). It can clearly be seen that, if the initial solution satisfies the boundary conditions appropriate for colliding waves, then so does the new solution. It can also

be seen that the transformation (12.1) is equivalent to adding to V the Stoyanov solution (10.29).

It may also be observed that the transformation (12.1) can be used to obtain the soliton solution of Ferrari and Ibañez (1987*b*) described in Section 10.4, using as seed the Khan–Penrose solution. Ferrari and Ibañez, however, originally obtained their solution using the soliton techniques mentioned in Section 12.6.

Another transformation which maps one colinear solution of (6.22) on another, whether or not they satisfy the boundary conditions, is given by

$$\begin{aligned} V &= bV_o \\ M &= (b^2 - 1) \log f' g' + \frac{1}{2}(b^2 - 1)U + b^2 M_o + \text{const} \end{aligned} \quad (12.2)$$

where b is an arbitrary constant and $e^{-U} = f(u) + g(v)$.

When $b \neq \pm 1$, it can be seen that the transformation (12.2) is singular on the boundaries of the interaction region where $f' = 0$ and $g' = 0$. It follows that the necessary boundary conditions cannot be satisfied for both solutions, if the same expressions for f and g are used. If the seed solution satisfies the boundary conditions as described in Chapter 7, with $U_o = -\log(f_o + g_o)$ and M_o containing the terms

$$M_o = \dots + k_{o1} \log(\tfrac{1}{2} - f_o) + k_{o2} \log(\tfrac{1}{2} - g_o) + \dots \quad (12.3)$$

with

$$f_o = \tfrac{1}{2} - (c_{o1}u)^{n_{o1}} + \dots, \quad g_o = \tfrac{1}{2} - (c_{o2}v)^{n_{o2}} + \dots \quad (12.4)$$

then the new solution containing the terms (7.10) will also satisfy the boundary conditions if f and g are replaced by

$$f = \tfrac{1}{2} - (c_1u)^{n_1} + \dots, \quad g = \tfrac{1}{2} - (c_2v)^{n_2} + \dots \quad (12.5)$$

where

$$1 - \frac{1}{n_1} = b^2 \left(1 - \frac{1}{n_{o1}}\right), \quad 1 - \frac{1}{n_2} = b^2 \left(1 - \frac{1}{n_{o2}}\right) \quad (12.6)$$

provided $n_1 \geq 2$ and $n_2 \geq 2$. These equations give the new expression for U in terms of the null coordinates u and v . It may be noted that the inequalities (12.6) provide strong constraints on the range of permissible values of the parameter b in (12.2).

It can thus be seen that this transformation with $b^2 \neq 1$ may be used to change the profile of the approaching waves. For example, it could

be used to derive the Szekeres solution described in Chapter 9 from the Khan–Penrose solution. The two transformations (12.1) and (12.2) may also be used in conjunction.

It may also be noted that the transformations for V in (12.1) and (12.2) may be restated in the form that, if a real Z_o is a solution of Ernst's equation (11.18), then

$$Z = (1 - t^2)^{a/2} (1 - z^2)^{a/2} Z_o \quad (12.7)$$

and

$$Z = Z_o^b \quad (12.8)$$

are also real solutions with arbitrary constants a and b , though as explained above, the possible values of b are constrained by the boundary conditions.

These results are, of course, implicitly contained in the general approach to colinear solutions described in Section 10.1. The point to note is that, when the approaching waves have constant aligned polarization so that Z and E are real and $W = 0$, the main field equations can be linearized. In this case a general superposition of solutions is possible, as illustrated in (10.16) together with (10.20). Further generating techniques for this case have been described by Kitchingham (1984).

In the more general case, however, Z and E are complex, and the field equations are essentially non-linear. In this situation, the generating techniques described in the remainder of this chapter are particularly significant.

12.2 Rotations and Ehlers transformations

Consider first the well known result that, if E_o is a solution of Ernst's equation (11.14), then

$$E = e^{i\alpha} E_o \quad (12.9)$$

where α is a constant, is also a solution.

In terms of the alternative function Z the equivalent result is that, if Z_o is a solution of (11.8), then another solution is given by

$$Z = \frac{\cos \frac{\alpha}{2} Z_o - i \sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2} - i \sin \frac{\alpha}{2} Z_o}. \quad (12.10)$$

By applying the coordinate rotation

$$x + iy = e^{i\alpha/2} (x_o + iy_o) \quad (12.11)$$

to the line element (11.12), it may be shown that this rotation is exactly equivalent to the transformation (12.9), indicating that this transformation may be simply interpreted as a global rotation of coordinates.

In terms of the metric functions of the line element (6.20), this rotation can be shown to be equivalent to the result that, if U_o , V_o , W_o and M_o are solutions of (6.22), then another solution is given by

$$\begin{aligned} U &= U_o \\ \tanh V &= \cos \alpha \tanh V_o + \sin \alpha \operatorname{sech} V_o \tanh W_o \\ \sinh W &= \cos \alpha \sinh W_o - \sin \alpha \sinh V_o \cosh W_o \\ M &= M_o. \end{aligned} \tag{12.12}$$

It may be noted that, in (12.12), the function M is unaltered. Thus the boundary conditions are not affected, and so neither are the functions $f(u)$ and $g(v)$. In fact, it is clear that (12.12) does not generate a new solution, but merely a rotation of the original.

It can also be shown that, if the seed solution is colinear so that we can put $W_o = 0$, then the transformation (12.12) reduces to the transformation suggested by Ray (1980) as corrected by Halilsoy (1981).

Another well known result is that, if Z_o is a solution of Ernst's equation (11.8), then

$$Z = \frac{Z_o}{1 - icZ_o} \tag{12.13}$$

where c is a real constant, is also a solution. This is loosely referred to as an Ehlers transformation (Ehlers, 1957).

Equivalently, if E_o is a solution of (11.14), then another solution is given by

$$1 + E = \frac{1 + E_o}{1 - \frac{ic}{2}(1 + E_o)}. \tag{12.14}$$

In terms of the metric functions in (6.20), we may start with the solutions U_o , V_o , W_o and M_o of (6.22). After integrating (11.21), it can then be shown that another solution is given by

$$\begin{aligned} U &= U_o \\ e^{2V} &= e^{2V_o} + 2ce^{V_o} \tanh W_o + c^2 \\ \sinh W &= \sinh W_o + c \cosh W_o e^{-V_o} \\ M &= M_o. \end{aligned} \tag{12.15}$$

Since the metric function M is unchanged, it follows that the boundary conditions for the new solution are satisfied if, and only if, they are satisfied for the seed solution.

The above transformation is also related to the well known result that, if Z_o is a solution of Ernst's equation (11.8), then another solution is given by

$$Z = Z_o + ib \quad (12.16)$$

where b is a real constant. Equivalently, if E_o is a solution of (11.14), then another solution is given by

$$1 - E = \frac{1 - E_o}{1 + \frac{ib}{2}(1 - E_o)}. \quad (12.17)$$

In terms of the metric functions in (6.20), this leads to the transformation that, if U_o , V_o , W_o and M_o are solutions of (6.22), then another solution is given by

$$\begin{aligned} U &= U_o \\ e^{-2V} &= e^{-2V_o} + 2be^{-V_o} \tanh W_o + b^2 \\ \sinh W &= \sinh W_o + b \cosh W_o e^{V_o} \\ M &= M_o. \end{aligned} \quad (12.18)$$

This can clearly be seen to be identical to (12.15) but with the sign of V interchanged.

All of the above transformations can be shown to be obtainable from a general rotation and rescaling of the coordinates in the x, y plane. This can be described by subjecting the Killing vectors ∂_x and ∂_y to any $SL(2, R)$ transformation

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}, \quad \text{where} \quad ad - bc = 1. \quad (12.19)$$

This leaves U and M invariant but, by considering the line element (11.6), the Ernst function Z can be shown to transform as

$$Z = i \frac{(aZ_o + ib)}{(cZ_o + id)}. \quad (12.20)$$

The transformation (12.20) clearly incorporates (12.10), (12.13) and (12.16) which include both pure rotations and those referred to as Ehlers transformations. When applied to metric functions on its own this transformation clearly has little physical significance. It follows that these transformations on their own can not be used to generate genuinely non-colinear solutions from colinear ones. However, they may be combined with some other method to obtain such solutions.

In particular these transformations may be applied to the potentials for the metric functions that will be described in Section 12.4. Strictly speaking it is only these transformations, applied in the potential space, that should be referred to as Ehlers transformations (Ehlers, 1957). It is then possible that a sequence both of coordinate transformations (12.19) and Ehlers transformations (12.20) in the potential space can be applied alternately to generate a further class of solutions with an infinite number of parameters.

12.3 Geroch transformations

Geroch (1971) has presented a general technique for generating solutions of Einstein's source-free equations from known solutions which possess a Killing vector ξ^μ . As originally presented, the method is to consider a vacuum space-time with metric $g_{\mu\nu}$, and to solve the equations

$$\begin{aligned}\nabla_\kappa G &= \epsilon_{\kappa\lambda\mu\nu} \xi^\lambda \nabla^\mu \xi^\nu \\ \nabla_{[\kappa} P_{\lambda]} &= \frac{1}{2} \epsilon_{\kappa\lambda\mu\nu} \nabla^\mu \xi^\nu \\ \nabla_{[\kappa} Q_{\lambda]} &= 2F \nabla_\kappa \xi_\lambda + G \epsilon_{\kappa\lambda\mu\nu} \nabla^\mu \xi^\nu\end{aligned}\tag{12.21}$$

for G , P_μ and Q_μ subject to the following conditions:

$$\xi^\mu \xi_\mu = F, \quad \xi^\mu P_\mu = G, \quad \xi^\mu Q_\mu = F^2 + G^2 - 1.\tag{12.22}$$

(A solution of these equations is known to exist.) Then, the basic result is that a new solution of the source-free equations, depending on an arbitrary real parameter α , is given by

$$g'_{\mu\nu} = F \tilde{F}^{-1} (g_{\mu\nu} - F^{-1} \xi_\mu \xi_\nu) + \tilde{F} \eta_\mu \eta_\nu\tag{12.23}$$

where

$$\begin{aligned}\tilde{F} &= F ((\cos \alpha - G \sin \alpha)^2 + F^2 \sin^2 \alpha)^{-1} \\ \eta_\mu &= \tilde{F}^{-1} \xi_\mu + 2 \sin \alpha \cos \alpha P_\mu - \sin^2 \alpha Q_\mu.\end{aligned}\tag{12.24}$$

It can be seen that F and G are respectively the norm and twist potential of the Killing field ξ^μ .

This technique determines a new solution with one extra parameter α . A second application of the method simply yields another member of the same one-parameter family of solutions. It may be observed that the new space-times also have the same Killing vector as the seed solution.

It may also be shown that these transformations include both the Ehlers transformation and the rotation described in previous sections.¹

The case when there exist two commuting Killing vectors, which is the case for colliding plane waves, has also been considered (Geroch, 1972). Clearly, the above technique may be applied using any linear combination of the Killing vectors. Successive applications of the method using different combinations of the Killing vectors yield metrics which depend on the order of the vectors used. The resulting class of solutions depends on two arbitrary functions, and retains the same pair of commuting Killing vectors. The algebraic structure of these transformations has been analysed by Geroch (1972).

The Geroch transformations, which relate one vacuum solution to another, in fact form an infinite-dimensional group. The first really useful realization of the Lie algebra of the Geroch group was formulated by Kinnersley and Chitre (1977, 1978*a,b*), who demonstrated the action of the infinitesimal elements of the group in terms of an infinite hierarchy of potentials. Kinnersley and Chitre, and also Hoenselaers, Kinnersley and Xanthopoulos (1979), exploited this approach to derive new stationary axisymmetric solutions. This formalism has been extended to space-times with two space-like Killing vectors by Kitchingham (1984), although he initially applied it only in the cosmological context.

Hauser and Ernst (1979, 1980) introduced a realization of the finite elements of the Geroch group. They also showed that the Kinnersley–Chitre transformations can be carried out by solving an appropriate homogeneous Hilbert problem.

A simpler realization $G_0(\Sigma_1)$ of the Geroch group of transformations which is specifically applied to colliding wave solutions has been found by Ernst, García-Díaz and Hauser (1987*b*). With this, colliding wave solutions with any desired number of parameters may be constructed.

When applying the original transformation (12.23) to colliding plane waves, it is convenient to start first with the Szekeres line element (6.20), in which the two Killing vectors are $\xi^{(1)\mu} = \delta_2^\mu$ and $\xi^{(2)\mu} = \delta_3^\mu$. It is then appropriate to consider a general Killing vector

$$\xi^\mu = a\delta_2^\mu + b\delta_3^\mu \quad (12.25)$$

where a and b are arbitrary constants. It follows immediately that

$$F = e^{-U}(a^2 e^V \cosh W - 2ab \sinh W + b^2 e^{-V} \cosh W). \quad (12.26)$$

¹ For a general introduction to Geroch and related transformations see Xanthopoulos (1985).

It is then possible to put two of the components of P_μ and Q_μ each equal to zero. Putting

$$P_\mu = (0, 0, P_2, P_3), \quad Q_\mu = (0, 0, Q_2, Q_3), \quad (12.27)$$

the remaining components of P_μ must then satisfy the equations

$$\begin{aligned} P_{2,u} &= -e^{-U} (ae^V (W_u - V_u \sinh W \cosh W) - b(U_u + V_u \cosh^2 W)) \\ P_{2,v} &= e^{-U} (ae^V (W_v - V_v \sinh W \cosh W) - b(U_v + V_v \cosh^2 W)) \\ P_{3,u} &= -e^{-U} (a(U_u - V_u \cosh^2 W) - be^{-V} (W_u + V_u \sinh W \cosh W)) \\ P_{3,v} &= e^{-U} (a(U_v - V_v \cosh^2 W) - be^{-V} (W_v + V_v \sinh W \cosh W)). \end{aligned} \quad (12.28)$$

These equations are automatically integrable in view of (6.22). It is then possible to put

$$G = aP_2 + bP_3 \quad (12.29)$$

and to choose the components of Q_μ such that

$$aQ_2 + bQ_3 = F^2 + G^2 - 1. \quad (12.30)$$

Equations (12.21) are then satisfied.

The new metric is then given by (12.23). The expressions for the new functions are rather involved. However, it is clear that

$$e^{-M} = ((\cos \alpha - G \sin \alpha)^2 + F^2 \sin^2 \alpha) e^{-M_0} \quad (12.31)$$

which indicates that the boundary conditions are satisfied for the new solution, if they were satisfied for the seed solution.

In the case when the initial solution has colinear polarization, this method can be considerably simplified. Such a situation was considered by Panov (1979*b*), and will be discussed further in Section 13.2.

12.4 The Neugebauer–Kramer involution

Chandrasekhar and Ferrari (1984) have shown that the main field equations for colliding plane waves can be written in the form of Ernst's equation (11.18) involving the complex function Z . In this case, the real and imaginary parts of this function, denoted by $Z = \chi + i\omega$, are the metric coefficients given by (11.2–4). Using these components, the main field equations can be written as

$$\left(\frac{(1-t^2)}{\chi} \chi_t \right)_{,t} - \left(\frac{(1-z^2)}{\chi} \chi_z \right)_{,z} + \frac{(1-t^2)}{\chi^2} \omega_t^2 - \frac{(1-z^2)}{\chi^2} \omega_z^2 = 0 \quad (12.32)$$

and

$$\left(\frac{(1-t^2)}{\chi^2}\omega_t\right),t - \left(\frac{(1-z^2)}{\chi^2}\omega_z\right),z = 0. \quad (12.33)$$

These are respectively the real and imaginary parts of (11.18).

The second of these equations immediately implies that there exists a function Φ such that

$$\Phi_z = \frac{(1-t^2)}{\chi^2}\omega_t, \quad \Phi_t = \frac{(1-z^2)}{\chi^2}\omega_z. \quad (12.34)$$

It follows that Φ must satisfy the equation

$$\left(\frac{\chi^2}{(1-z^2)}\Phi_t\right),t - \left(\frac{\chi^2}{(1-t^2)}\Phi_z\right),z = 0. \quad (12.35)$$

At this point, it is convenient to introduce a new function Ψ defined by

$$\Psi = \sqrt{1-t^2}\sqrt{1-z^2}\chi^{-1}. \quad (12.36)$$

From (11.1) and (10.11), it may be seen that $-\Psi$ is the coefficient of dx^2 in the line element (11.4).

With this definition and (12.34), the equations (12.32) and (12.35) can now be rewritten in the form

$$\left(\frac{(1-t^2)}{\Psi}\Psi_t\right),t - \left(\frac{(1-z^2)}{\Psi}\Psi_z\right),z + \frac{(1-t^2)}{\Psi^2}\Phi_t^2 - \frac{(1-z^2)}{\Psi^2}\Phi_z^2 = 0 \quad (12.37)$$

and

$$\left(\frac{(1-t^2)}{\Psi^2}\Phi_t\right),t - \left(\frac{(1-z^2)}{\Psi^2}\Phi_z\right),z = 0. \quad (12.38)$$

These equations may be seen to be identical to (12.32) and (12.33) with Ψ and Φ replacing χ and ω . Thus, if $Z_o = \chi + i\omega$ is a solution of (11.18), then another solution is given by

$$Z = \Psi + i\Phi \quad (12.39)$$

where Ψ is given by (12.36), and Φ is chosen to satisfy (12.34). This correspondence is known as the Neugebauer–Kramer involution. It is known to map real metrics to real metrics only for space-times with two space-like Killing vectors. In modern terminology, it is clear that the equations (12.34) provide an auto-Bäcklund transformation for the Ernst equation (11.18).

In order to obtain the remaining metric function M it is necessary to integrate equations (7.9), which may be rewritten in the form

$$\begin{aligned} S_f &= -\frac{1}{2} \frac{(f+g)}{\chi^2} (\chi_f^2 + \omega_f^2) \\ S_g &= -\frac{1}{2} \frac{(f+g)}{\chi^2} (\chi_g^2 + \omega_g^2) \end{aligned} \quad (12.40)$$

which is equivalent to (11.21). In terms of the alternative functions Ψ and Φ these become

$$\begin{aligned} S_f &= -\frac{1}{2(f+g)} + \frac{\Psi_f}{\Psi} - \frac{1}{2} \frac{(f+g)}{\Psi^2} (\Psi_f^2 + \Phi_f^2) \\ S_g &= -\frac{1}{2(f+g)} + \frac{\Psi_g}{\Psi} - \frac{1}{2} \frac{(f+g)}{\Psi^2} (\Psi_g^2 + \Phi_g^2). \end{aligned} \quad (12.41)$$

It follows from this that, for any solution with $Z_o = \chi + i\omega$ and M_o , there exists another solution with $Z = \Psi + i\Phi = Z_o$ and M where

$$M = M_o + \frac{1}{2} U_o + \log \chi. \quad (12.42)$$

This immediately indicates that, if the initial solution satisfies the boundary conditions for colliding plane waves, then so also does the new solution, provided

$$U = U_o. \quad (12.43)$$

A second application of this technique only leads back to the original solution, apart from an arbitrary complex constant. The presence of such a constant has already been considered in Section 12.2 following equation (12.16).

It may be noted in passing that, in the colinear case in which $\omega = 0$ and $\Phi = 0$, this generation technique is equivalent to putting $V = U_o - V_o$ which is effectively contained in (12.1) but with a change in the sign of V . This, for example, would generate the Ferrari–Ibañez solution from the Khan–Penrose solution.

It is of interest to observe that, for stationary axisymmetric space-times, the main field equations can be written in the form of Ernst's equation (11.8), whose solution $Z = \Psi + i\Phi$ is considered as a complex potential for the metric functions χ and ω that are obtained from (12.36) and (12.34). In contrast, for colliding plane waves, both the potential (12.39) and the metric function (11.5) satisfy the same equation. Thus, in this case, the Neugebauer–Kramer involution is equivalent to a generation

technique in which any solution has a dual solution in which the metric functions and the associated potentials are interchanged.

12.5 A combined transformation

In a recent paper, Halilsoy (1988*b*) has considered a technique which he applied initially in the context of colliding shock electromagnetic waves (Halilsoy, 1988*a*). This can be described as follows.

If V_o is a solution of the vacuum equations (9.3), (10.2) or (10.13) corresponding to a collision of colinear gravitational waves (with $W_o = 0$), then a noncolinear solution is given in terms of the function E satisfying (11.14) by

$$E = P e^{i\theta} \quad (12.44)$$

where

$$P^2 = \frac{\cosh aV_o - \cos \alpha}{\cosh aV_o + \cos \alpha} \quad (12.45)$$

$$\tan \theta = -\tan \alpha \coth aV_o$$

where α is an arbitrary constant. The original solution is obtained when $\alpha = 0$ and $a = 1$.

In terms of the alternative function Z , the original solution may be described by a real function Z_o and the new solution satisfying (11.8) is then given by

$$Z = \frac{i \tan \frac{\alpha}{2} + Z_o^a}{1 - i \tan \frac{\alpha}{2} Z_o^a}. \quad (12.46)$$

This can thus be seen to be a transformation of the type (12.2) followed by an Ehlers transformation in a slightly more general form than (12.13).

In terms of the metric functions of the Szekeres line element (6.20), the original solution is described by the three functions U_o , V_o and M_o which are each functions of f_o and g_o where, as in (7.8), it is convenient to put

$$e^{-U_o} = f_o + g_o, \quad e^{-M_o} = \frac{f_o' g_o'}{\sqrt{f_o + g_o}} e^{-S_o} \quad (12.47)$$

and, as in (12.4),

$$f_o = \frac{1}{2} - (c_{o1}u)^{n_{o1}} + \dots, \quad g_o = \frac{1}{2} - (c_{o2}v)^{n_{o2}} + \dots \quad (12.48)$$

The new solution (12.44) is given by

$$\begin{aligned} \tanh V &= \cos \alpha \tanh aV_o \\ \sinh W &= \tan \alpha \cosh aV_o. \end{aligned} \quad (12.49)$$

With this, equations (7.7) can be integrated to give

$$M = (a^2 - 1) \log f'_o g'_o + \frac{1}{2}(a^2 - 1)U_o + a^2 M_o + \text{const} \quad (12.50)$$

which is similar to (12.2). Now, using the same arguments as in Section 12.1, the new solution will only satisfy the appropriate boundary conditions if the forms of the functions f_o and g_o are modified. Accordingly we now put

$$e^{-U} = f + g, \quad e^{-M} = \frac{f'g'}{\sqrt{f+g}} e^{-S}. \quad (12.51)$$

The parameters f and g are now expressed in terms of the null coordinates u and v in the form

$$f = \frac{1}{2} - (c_1 u)^{n_1} + \dots, \quad g = \frac{1}{2} - (c_2 v)^{n_2} + \dots \quad (12.52)$$

where

$$1 - \frac{1}{n_1} = a^2 \left(1 - \frac{1}{n_{o1}}\right), \quad 1 - \frac{1}{n_2} = a^2 \left(1 - \frac{1}{n_{o2}}\right). \quad (12.53)$$

With this restriction, the boundary conditions are satisfied, and

$$S = a^2 S_o. \quad (12.54)$$

If, in the original solution, the functions f_o , g_o and M_o have been chosen such that the terms $f'_o g'_o$ in (12.47) have already cancelled the terms $k_{o1} \log(\frac{1}{2} - f_o)$ and $k_{o2} \log(\frac{1}{2} - g_o)$ in S_o , then the new solution for M will be given by

$$M = \frac{1}{2}(a^2 - 1)U_o + a^2 M_o. \quad (12.55)$$

The method described in this section has been used by Halilsoy (1988*b*) to obtain a class of solutions which appears to generalize the Szekeres solutions. However, since it has been obtained by a real transformation followed by an Ehlers transformation, as described at the end of Section 12.2 it is essentially simply a rotation and rescaling of the Szekeres solutions. It does not therefore describe the collision of genuinely non-aligned gravitational waves.

12.6 Other methods

Within the context of stationary axisymmetric space-times, many techniques have been developed in recent years by which new solutions may

be obtained from initial vacuum solutions.² Most of these methods can be applied also to the colliding plane wave situation. The main difference being that, in this case, the solutions of Ernst's equation may themselves either be metric functions or potentials for those functions as described in Section 12.4.

The other difference relates to the change of boundary conditions. For stationary axisymmetric solutions, it is usual to impose the condition that the space-times are asymptotically flat, and techniques for generating new solutions of Ernst's equation in which the space-time satisfies this condition have been developed. One of the most general of these is that of Cosgrove (1977), which generalizes the Tomimatsu–Sato solutions to arbitrary continuous deformation parameter δ . However, for colliding plane waves, the boundary conditions are of a totally different character, and many of the solutions of Ernst's equation that are appropriate for axisymmetric fields, such as the Tomimatsu–Sato solutions, are now found to be inconsistent with the boundary conditions for colliding plane waves.

Hassan, Feinstein and Manko (1990) have adapted a generating technique of Gutsunaev and Manko (1988) for stationary axisymmetric space-times to derive solutions which describe the interaction following the collision of plane gravitational waves. This technique generates solutions in which the approaching waves have variable polarization from a 'seed' solution V_o of the equation (10.13) for the colinear case. A solution of the Ernst equation (11.17) is then given by

$$Z = e^{V_o} \frac{(1 - \alpha^2 ab)t + i\alpha(a + b)z - (1 + i\alpha a)(1 - i\alpha b)}{(1 - \alpha^2 ab)t + i\alpha(a + b)z + (1 + i\alpha a)(1 - i\alpha b)} \quad (12.56)$$

where α is a constant and the functions $a(t, z)$ and $b(t, z)$ are obtained by integrating the equations

$$\begin{aligned} (\log a)_{,t} &= \frac{1}{t - z} [(tz - 1)V_{o,t} + (1 - z^2)V_{o,z}] \\ (\log a)_{,z} &= \frac{1}{t - z} [(1 - t^2)V_{o,t} + (tz - 1)V_{o,z}] \\ (\log b)_{,t} &= \frac{1}{t + z} [(tz + 1)V_{o,t} + (1 - z^2)V_{o,z}] \\ (\log b)_{,z} &= \frac{1}{t + z} [(1 - t^2)V_{o,t} + (tz + 1)V_{o,z}]. \end{aligned} \quad (12.57)$$

The boundary conditions then have to be applied.

² For recent reviews see Kramer (1987), Dodd (1990) and the conference proceedings edited by Hoenselaers and Dietz (1984).

One particularly important technique that has been developed in recent years is the inverse scattering method of Belinskii and Zakharov (1978, 1979) which had previously been developed as a soliton technique. This method has been applied by Carr and Verdaguer (1983) to plane-symmetric space-times, and their results have been interpreted in terms of cosmological gravitational waves. The relation between this and other techniques has been analysed by Letelier (1987).

The first solution for colliding plane gravitational waves that was obtained using the inverse scattering method was that of Ferrari and Ibañez (1987*b*), who obtained the solution described in Section 10.4 using the method in the form developed by Carr and Verdaguer (1983). In this paper they took as seed metric the Kasner solution (10.26) which is equivalent to the Stoyanov solution (10.29). It may therefore be noted that, although the new solution satisfies the boundary conditions for colliding plane waves, the seed solution in this case does not. Non-aligned solutions have also been obtained using this method by Ferrari, Ibañez and Bruni (1987*b*).

Another solution-generating technique that has been developed in recent years is that associated with Bäcklund transformations. These were first applied to general relativity independently by Harrison (1978) and Neugebauer (1979). Bäcklund transformations can also be described as double-Harrison transformations, or as quadruple-Neugebauer transformations. The explicit relation between this technique and other methods has been discussed by Cosgrove (1980, 1982).

The effect of a double-Harrison transformation can be deduced by any number of competing techniques. Hauser and Ernst (1979, 1980) developed a method which involves solving a homogeneous Hilbert problem. Cosgrove (1981) has shown how Bäcklund transformations can be conveniently expressed in this formalism. The method has been applied to colliding wave problems by Ernst, García-Díaz and Hauser (1988) (see also Ernst, 1988), and a class of new colliding wave solutions has been obtained.

It has been observed by Witten (1979) that Einstein's vacuum equations for stationary, axisymmetric space-times are equivalent to a form of the self-dual Yang-Mills equations. He therefore suggested that it should be possible to generate solutions of Einstein's equations by using twistor methods to construct self-dual Yang-Mills fields with appropriate symmetries. Such an approach to stationary axisymmetric space-times has subsequently been developed by Ward (1983), Woodhouse (1987) and also by Woodhouse and Mason (1988), who showed how this approach relates to the other solution-generating techniques mentioned above. It follows that this approach may also be applied for other types of space-

times which contain two commuting Killing vectors. Woodhouse (1989) has applied these techniques to generate cylindrical gravitational waves. Twistor techniques may also be developed to generate colliding plane wave solutions.

Any of the above techniques may be used to generate new solutions of Ernst's equation. Most have been applied mainly to stationary axisymmetric space-times. They may, however, also be used to generate solutions of Ernst's equation that can be considered as possible to colliding plane wave solutions. The difficulty is then to find under what circumstances the appropriate boundary conditions are satisfied. Whether or not a new solution in region IV satisfies these conditions will generally depend both on the particular technique that is employed and also on the initial seed solution.